

EXISTENCE OF CYLINDRICALLY SYMMETRIC GROUND STATES TO A NONLINEAR CURL-CURL EQUATION WITH NON-CONSTANT COEFFICIENTS

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ABSTRACT. We consider the nonlinear curl-curl problem $\nabla \times \nabla \times U + V(x)U = f(x, |U|^2)U$ in \mathbb{R}^3 related to the nonlinear Maxwell equations with Kerr-type nonlinear material laws. We prove the existence of a symmetric ground-state type solution for a bounded, cylindrically symmetric coefficient V and subcritical cylindrically symmetric nonlinearity f . The new existence result extends the class of problems for which ground-state type solutions are known. It is based on compactness properties of symmetric functions [11, 12], new rearrangement type inequalities from [6] and the recent extension of the Nehari-manifold technique from [18].

1. INTRODUCTION

We consider the system

$$(1.1) \quad \nabla \times \nabla \times U + V(x)U = f(x, |U|^2)U \text{ in } \mathbb{R}^3$$

where $V \in L^\infty(\mathbb{R}^3)$ and $f : \mathbb{R}^3 \times [0, \infty) \rightarrow [0, \infty)$ is a non-negative Carathéodory function growing at infinity with a power at most $\frac{p-1}{2}$ for $p \in (1, 5)$. The particular feature of (1.1) is the curl-curl operator. It arises in specific models for standing waves in Maxwell's equations with Kerr-type nonlinear material laws where $f(x, |U|^2)U = \Gamma(x)|U|^2U$. For a detailed physical motivation of (1.1) see [2].

We look for \mathbb{R}^3 -valued weak solutions U in a cone $K_{4,1}$ of functions with suitable symmetries and $U \in L^2(\mathbb{R}^3) \cap L^{p+1}(\mathbb{R}^3)$, $\nabla \times U \in L^2(\mathbb{R}^3)$. The condition that 0 lies below the spectrum of curl curl + $V(x)$ allows us to find ground-state type critical points of a functional $J(u) = \frac{1}{2}\|u\|^2 - I(u)$, cf. (1.4), restricted to the so-called Nehari-manifold. The basic idea of applying symmetrizations to minimizing sequences on the Nehari-manifold goes back to Stuart [17] in the context of the stationary nonlinear Schrödinger equation. Compared to [17] the assumptions on the nonlinearity f can be substantially weakened beyond the classical Ambrosetti-Rabinowitz condition. This is based on three important ingredients:

- the recent extension of the Nehari-manifold method due to Szulkin and Weth [18],
- the weak sequential continuity of functionals $I(u)$ and $I'(u)[u]$ on $K_{4,1}$ due to compactness properties of symmetric functions by Lions [11, 12],
- new rearrangement inequalities for general nonlinearities due to Brock [6].

Using the combination of these ingredients our main result of Theorem 1 substantially extends the known results on the existence of ground-state type solutions for (1.1).

Benci, Fortunato [5] and Azzollini, Benci, D'Aprile, Fortunato in [1] were among the first to consider the constant coefficient case of (1.1) with $V \equiv 0$. Their method was based on cylindrical symmetries of the vector-fields U , cf. [8] for a different class of symmetries. The case where

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$f(x, |U|^2)U = \Gamma(x)|U|^{p-1}U$ with periodic coefficients V and Γ has been treated in [2]. In [14] Mederski considered (1.1) where $f(x, |U|^2)U$ is replaced by, e.g., $\Gamma(x)g(U)$ with $\Gamma > 0$ periodic and bounded, $V \leq 0$, $V \in L^{\frac{p+1}{p-1}}(\mathbb{R}^3) \cap L^{\frac{q+1}{q-1}}(\mathbb{R}^3)$ and $g(U) \sim |u|^{p-1}U$ if $|U| \gg 1$ and $g(U) \sim |U|^{q-1}u$ if $|U| \ll 1$ for $1 < p < 5 < q$. A remarkable feature of Mederski's work is that (1.1) can be treated without assuming special symmetries of the field U . The nonlinear curl-curl problem on bounded domains with the boundary condition $\nu \times U = 0$ has been elaborated in [3, 4].

An important feature of [1] is the use of cylindrically symmetric ansatz functions for U . Here we make a slightly different ansatz of the form

$$(1.2) \quad U(x) = u(r, z) \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \text{ where } r = \sqrt{x_1^2 + x_2^2}, \quad z = x_3.$$

Moreover, we assume cylindrically symmetric coefficients $V(x) = V(r, z)$, $f(x, |U|^2) = f(r, z, |U|^2)$. For U of the form (1.2) we see that $\operatorname{div} U = 0$, and hence (1.1) reduces to the scalar equation

$$(1.3) \quad -\frac{1}{r^3} \frac{\partial}{\partial r} \left(r^3 \frac{\partial u}{\partial r} \right) - \frac{\partial^2 u}{\partial z^2} + V(r, z)u = f(r, z, r^2 u^2)u \quad \text{for } (r, z) \in \Omega := (0, \infty) \times \mathbb{R}.$$

It turns out that a suitable space to consider (1.3) is given by

$$H_{\text{cyl}}^1(r^3 dr dz) := \left\{ v : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R} : v, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial z} \in L_{\text{cyl}}^2(r^3 dr dz) \right\},$$

$$L_{\text{cyl}}^2(r^3 dr dz) := \left\{ v : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R} : \int_{\Omega} v(r, z)^2 r^3 d(r, z) < \infty \right\},$$

cf. Section 2 for more details on these spaces. Weak solutions of (1.3) arise as critical points of the functional

$$(1.4) \quad J(u) = \frac{1}{2} \int_{\Omega} (|\nabla_{r,z} u|^2 + V(r, z)u^2) r^3 d(r, z) - \int_{\Omega} \frac{1}{2r^2} F(r, z, r^2 u^2) r^3 d(r, z), \quad u \in H_{\text{cyl}}^1(r^3 dr dz),$$

where $F(r, z, t) := \int_0^t f(r, z, s) ds$ and $\nabla_{r,z} := \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial z} \right)$. A ground state u of (1.3) is defined as a weak solution of (1.3) in the Nehari-manifold

$$M := \left\{ v \in H_{\text{cyl}}^1(r^3 dr dz) \setminus \{0\} : \int_{\Omega} (|\nabla_{r,z} v|^2 + V(r, z)v^2) r^3 d(r, z) = \int_{\Omega} f(r, z, r^2 v^2) v^2 r^3 d(r, z) \right\}$$

such that

$$J(u) = \inf_{v \in M} J(v),$$

see the classical papers [15], [16]. We find ground states of (1.3) under additional assumptions on V and f . To state these assumptions we need the notion of Steiner-symmetrization, cf. Chapter 3 in [10]. The Steiner-symmetrization (also called symmetric-decreasing rearrangement) of a cylindrical function $g = g(r, z)$ with respect to z is denoted by g^* . We say that g is *Steiner-symmetric* if g coincides with its Steiner-symmetrization with respect to z , keeping the r -variable fixed. A function $h \in L^\infty(\Omega)$ is *reversed Steiner-symmetric* if $(\operatorname{ess\,sup} h - h)^* = \operatorname{ess\,sup} h - h$ holds true.

Now we can state our assumptions on f .

- (i) $f : \Omega \times [0, \infty) \rightarrow \mathbb{R}$ is a Carathéodory function with $0 \leq f(r, z, s) \leq c(1 + s^{\frac{p-1}{2}})$ for some $c > 0$ and $p \in (1, 5)$,
- (ii) $f(r, z, s) = o(1)$ as $s \rightarrow 0$ uniformly in $r, z \in [0, \infty) \times \mathbb{R}$,
- (iii) $f(r, z, s)$ strictly increasing in $s \in [0, \infty)$,

- (iv) $\frac{F(r,z,s)}{s} \rightarrow \infty$ as $s \rightarrow \infty$ uniformly in $r, z \in [0, \infty) \times \mathbb{R}$,
(v) for all $r \in [0, \infty)$, $s \geq 0$ and $\sigma > 0$ the function

$$\varphi_\sigma(r, z, s) := f(r, z, (s + \sigma)^2)(s + \sigma)^2 - f(r, z, s^2)s^2$$

is symmetrically nonincreasing in z .

Conditions (ii)–(iv) are inspired by the work of Szulkin and Weth [18]. Namely, if we translate (ii)–(iv) into conditions for $\tilde{f}(r, z, s) := f(r, z, r^2 s^2)s$ then they become identical to (ii)–(iv) of Theorem 20 from [18]. Condition (v) is used to prove the rearrangement inequality of Lemma 9 and it is due to Brock [6].

Next we state our main result.

Theorem 1. *Let $V \in L^\infty(\Omega)$ be reversed Steiner-symmetric such that the map*

$$(1.5) \quad \|\cdot\| : H_{\text{cyl}}^1(r^3 dr dz) \rightarrow \mathbb{R}; u \mapsto \left(\int_{\Omega} (|\nabla_{r,z} u|^2 + V(r, z)u^2) r^3 d(r, z) \right)^{\frac{1}{2}}$$

is an equivalent norm to $\|\cdot\|_{H_{\text{cyl}}^1(r^3 dr dz)}$. Additionally, let f satisfy the assumptions (i)–(v). Then (1.3) has a ground state $u \in H_{\text{cyl}}^1(r^3 dr dz)$ which is symmetric about $\{z = 0\}$.

Remarks: (1) The assumption of norm-equivalence is for instance satisfied if $V \geq 0$ and $\inf_{B_R^c} V > 0$ for some $R > 0$, where $B_R^c := \{(r, z) \in \Omega : r^2 + z^2 > R^2\}$. For the reader's convenience the proof based on Poincaré's inequality is given in the Appendix. Since Poincaré's inequality is applicable for domains bounded in one direction we can weaken $\inf_{B_R^c} V > 0$ to $\inf_{S^c} V > 0$ for strips $S = [0, \infty) \times [0, \rho]$ with $\rho > 0$ or $S = [r_0, r_1] \times [0, \infty)$ with $0 \leq r_0 < r_1 < \infty$.

(2) The conditions on f are satisfied if for instance $f(r, z, s) = \Gamma(r, z)|s|^{\frac{p-2}{2}}s$ where $\Gamma \in L^\infty(\Omega)$ is Steiner-symmetric, $\text{ess inf}_{\Omega} \Gamma > 0$ and $p \in (1, 5)$. This choice of f corresponds to the equation $\nabla \times \nabla \times U + V(r, z)U = \Gamma(r, z)|U|^{p-1}U$ in \mathbb{R}^3 . Another possible choice is $f(r, z, s) = \Gamma(r, z) \log(1 + s)$ where again $\Gamma \in L^\infty(\Omega)$ is Steiner-symmetric and $\text{ess inf}_{\Omega} \Gamma > 0$. This nonlinearity appeared for instance in [13] and it does not satisfy the classical Ambrosetti-Rabinowitz condition.

The paper is structured as follows: In Section 2 we give details on the variational formulation of problem (1.3) and prove pointwise decay estimates of Steiner-symmetric functions in $H_{\text{cyl}}^1(r^3 dr dz)$. In Section 3 we give the proof of Theorem 1, and in the Appendix we show an example for the potential V satisfying the equivalent-norm-assumption of Theorem 1.

2. VARIATIONAL FORMULATION, DECAY ESTIMATES, REARRANGEMENTS

Let us consider some properties of the space $H_{\text{cyl}}^1(r^3 dr dz)$. First, for U of the form (1.2) we have that $U \in H^1(\mathbb{R}^3)$ if and only if $u \in H_{\text{cyl}}^1(r^3 dr dz)$. A norm on $H_{\text{cyl}}^1(r^3 dr dz)$ is given by

$$\|u\|_{H_{\text{cyl}}^1(r^3 dr dz)} := \left(\int_{\Omega} (|\nabla_{r,z} u|^2 + u^2) r^3 d(r, z) \right)^{\frac{1}{2}}.$$

Notice that the space $H_{\text{cyl}}^1(r^3 dr dz)$ behaves like a Sobolev-space in dimension 5. Next we show a useful embedding property. For this we need the following Sobolev and Lebesgue spaces in dimension 3 together with their canonical norms:

$$H_{\text{cyl}}^1(r dr dz) := \left\{ v : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R} : v, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial z} \in L_{\text{cyl}}^2(r dr dz) \right\},$$

$$L_{\text{cyl}}^q(rdrdz) := \left\{ v: (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R} : \int_{\Omega} |v(r, z)|^q r dr dz < \infty \right\} \text{ for } q \in [1, \infty).$$

Lemma 2. For $u \in H_{\text{cyl}}^1(r^3 drdz)$ Hardy's inequality holds

$$(2.1) \quad \int_{\Omega} \frac{u^2}{r^2} r^3 dr dz \leq C_H \int_{\Omega} \left(\left(\frac{\partial u}{\partial r} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 \right) r^3 dr dz.$$

Moreover, if $u \in H_{\text{cyl}}^1(r^3 drdz)$ then $ru \in H_{\text{cyl}}^1(rdrdz)$ and there is a constant $C > 0$ such that for $2 \leq q \leq 6$

$$(2.2) \quad \|ru\|_{H_{\text{cyl}}^1(rdrdz)}, \|ru\|_{L_{\text{cyl}}^q(rdrdz)} \leq C \|u\|_{H_{\text{cyl}}^1(r^3 drdz)}$$

Proof. Hardy's inequality (2.1) is given in Lemma 9 (i) in [2]. For $u \in H_{\text{cyl}}^1(r^3 drdz)$ we have $ru, \frac{\partial}{\partial z}(ru), r \frac{\partial u}{\partial r} \in L_{\text{cyl}}^2(rdrdz)$ and by (2.1) also $u \in L_{\text{cyl}}^2(rdrdz)$. Since $\frac{\partial}{\partial r}(ru) = r \frac{\partial u}{\partial r} + u$ we conclude altogether $ru \in H_{\text{cyl}}^1(rdrdz)$. By the Sobolev embedding in three dimensions this implies $ru \in L^q(rdrdz)$ for $q \in [2, 6]$ and (2.1) yields

$$(2.3) \quad \begin{aligned} \|ru\|_{H_{\text{cyl}}^1(rdrdz)}^2 &= \int_{\Omega} (|\nabla_{r,z}(ru)|^2 + r^2 u^2) r dr dz \\ &\leq 2 \int_{\Omega} \left(\left(r \frac{\partial u}{\partial z} \right)^2 + \left(r \frac{\partial u}{\partial r} \right)^2 + u^2 + r^2 u^2 \right) r dr dz \leq \tilde{C} \|u\|_{H_{\text{cyl}}^1(r^3 drdz)}^2. \end{aligned}$$

□

Next we show that the functional J from the introduction as well as the functional in the definition of the Nehari-manifold are well-defined.

Lemma 3. There is a constant $C > 0$ such that

$$(2.4) \quad \int_{\Omega} f(r, z, r^2 u^2) u^2 r^3 dr dz, \int_{\Omega} \frac{1}{2r^2} F(r, z, r^2 u^2) r^3 dr dz \leq C \left(\|u\|_{H_{\text{cyl}}^1(r^3 drdz)}^2 + \|u\|_{H_{\text{cyl}}^1(r^3 drdz)}^{p+1} \right)$$

for all $u \in H_{\text{cyl}}^1(r^3 drdz)$.

Proof. Clearly assumption (i) and (ii) show that for every $\epsilon > 0$ there is $C_{\epsilon} > 0$ such that

$$0 \leq f(r, z, s) \leq \epsilon + C_{\epsilon} s^{\frac{p-1}{2}}.$$

Hence

$$(2.5) \quad 0 \leq f(r, z, r^2 u^2) u^2 r^3 \leq (\epsilon r^2 u^2 + C_{\epsilon} |ru|^{p+1}) r,$$

$$(2.6) \quad 0 \leq \frac{1}{2r^2} F(r, z, r^2 u^2) r^3 \leq (\epsilon r^2 u^2 + \tilde{C}_{\epsilon} |ru|^{p+1}) r.$$

Due to (2.2) this implies the claim. □

In order to find critical points of J we need uniform decay estimates of Steiner-symmetric functions in $H_{\text{cyl}}^1(r^3 drdz)$. These estimates are given in [12] in much more generality but for the sake of completeness we give them here together with the simple proof. We start with a well-known fact concerning radially symmetric functions and afterwards extend the result to cylindrically symmetric functions. Let

$$H_{\text{rad}}^1(\mathbb{R}^n) := \{u \in H^1(\mathbb{R}^n) : u \text{ is radially symmetric}\}.$$

Lemma 4. (see [12]) *Let $n \geq 2$. Then there is a constant $C > 0$ such that*

$$|u(x)| \leq C \|\nabla u\|_{L^2(\mathbb{R}^n)}^{1/2} \|u\|_{L^2(\mathbb{R}^n)}^{1/2} |x|^{-(n-1)/2} \text{ for almost all } x \in \mathbb{R}^n \text{ and all } u \in H_{\text{rad}}^1(\mathbb{R}^n).$$

Proof. By density it is sufficient to prove the estimate for $u \in H_{\text{rad}}^1(\mathbb{R}^n) \cap C_c^\infty(\mathbb{R}^n)$. Let $r := |x|$. Then

$$\frac{d}{dr} \left(r^{n-1} |u|^2 \right) = (n-1) r^{n-2} |u|^2 + r^{n-1} 2u \frac{\partial u}{\partial r} \geq -2 |u| \left| \frac{\partial u}{\partial r} \right| r^{n-1}.$$

Integrating from r to ∞ and expanding the domain of integration to all of \mathbb{R}^n yields

$$\square \quad r^{n-1} |u(x)|^2 \leq C \int_{\mathbb{R}^n} |u| |\nabla u| dy \leq C \|\nabla u\|_{L^2(\mathbb{R}^n)} \|u\|_{L^2(\mathbb{R}^n)}.$$

Now we give an extension of Lemma 4 to cylindrically symmetric functions which are Steiner-symmetric in the non-radial component. We make use of the following notation: Let $t \in \mathbb{N}_{\geq 2}$ and $s \in \mathbb{N}$ such that $n = t + s$. We write points in \mathbb{R}^n as (x, y) with $x \in \mathbb{R}^t$ and $y = (y_1, \dots, y_s) \in \mathbb{R}^s$. Furthermore, let

$$K_{t,s} := \left\{ u \in H^1(\mathbb{R}^n) \text{ s.t. } \begin{cases} u(\cdot, y) & \text{is a radially symmetric function for every } y \in \mathbb{R}^s \text{ and} \\ u(x, \cdot) & \text{is Steiner-symmetric w.r.t. } y_i, i = 1, \dots, s, \text{ for every } x \in \mathbb{R}^t \end{cases} \right\}.$$

In particular, if $u \in K_{t,s}$ then necessarily $u \geq 0$. In this setting we have the following extension of Lemma 4.

Lemma 5. (see [12]) *There is a constant $C > 0$ such that*

$$0 \leq u(x, y) \leq C \|\nabla_x u\|_{L^2(\mathbb{R}^n)}^{1/2} \|u\|_{L^2(\mathbb{R}^n)}^{1/2} |x|^{-(t-1)/2} |y_1 \cdots y_s|^{-1/2} \text{ for almost all } (x, y) \in \mathbb{R}^n \text{ and all } u \in K_{t,s}.$$

Proof. Let $u \in K_{t,s}$ and fix $y \in \mathbb{R}^s$. W.l.o.g. let $y_i > 0$ for all $i = 1, \dots, s$. We define

$$v(x) := \int_0^{y_1} \cdots \int_0^{y_s} u(x, z) dz \text{ for } x \in \mathbb{R}^t.$$

By Hölder's inequality we obtain $v^2(x) \leq y_1 \cdots y_s \int_0^{y_1} \cdots \int_0^{y_s} u^2(x, z) dz$, i.e.,

$$(2.7) \quad \|v\|_{L^2(\mathbb{R}^t)} \leq (y_1 \cdots y_s)^{1/2} \|u\|_{L^2(\mathbb{R}^n)}.$$

In the same manner we receive

$$(2.8) \quad \|\nabla v\|_{L^2(\mathbb{R}^t)} \leq (y_1 \cdots y_s)^{1/2} \|\nabla_x u\|_{L^2(\mathbb{R}^n)}.$$

Since $v: \mathbb{R}^t \rightarrow \mathbb{R}$ is radially symmetric we can apply Lemma 4 and get from (2.7) and (2.8)

$$(2.9) \quad 0 \leq v(x) \leq C \|\nabla v\|_{L^2(\mathbb{R}^t)}^{1/2} \|v\|_{L^2(\mathbb{R}^t)}^{1/2} |x|^{-(t-1)/2} \leq C (y_1 \cdots y_s)^{1/2} \|\nabla_x u\|_{L^2(\mathbb{R}^n)}^{1/2} \|u\|_{L^2(\mathbb{R}^n)}^{1/2} |x|^{-(t-1)/2}.$$

Due to the monotonicity-property in y -direction we also have $v(x) \geq y_1 \cdots y_s u(x, y)$ and thus (2.9) gives the desired inequality. \square

We prove three additional lemmas which are used in the next section.

Lemma 6. *The set $K_{t,s}$ is a weakly closed cone in $H^1(\mathbb{R}^n)$.*

Proof. Take a sequence $(u_k)_{k \in \mathbb{N}} \subset K_{t,s}$ such that $u_k \rightharpoonup u \in H^1(\mathbb{R}^n)$ as $k \rightarrow \infty$. By the Sobolev embedding on bounded domains we deduce that a subsequence of u_k converges pointwise almost everywhere on \mathbb{R}^n to u . Since every u_k enjoys the radial symmetry in the first component and the non-increasing property in the second variable, the pointwise convergence implies that also u enjoys these properties, i.e., $u \in K_{t,s}$. \square

Lemma 7. *The functionals*

$$I(v) = \int_{\Omega} \frac{1}{2r^2} F(r, z, r^2 v^2) r^3 d(r, z), \quad I'(v)[v] = \int_{\Omega} f(r, z, r^2 v^2) v^2 r^3 d(r, z)$$

are weakly sequentially continuous on the set $K_{4,1} \subset H_{\text{cyl}}^1(r^3 dr dz)$.

Remark: In the proof we use twice the following principle: if $S \subset \mathbb{R}^m$ is a set of finite measure and $w_k : S \rightarrow \mathbb{R}$ a sequence of measurable functions such that $\|w_k\|_{L^q(S)} \leq C$ and $w_k \rightarrow w$ pointwise a.e. as $k \rightarrow \infty$ then $\|w_k - w\|_{L^q(S)} \rightarrow 0$ as $k \rightarrow \infty$ for $1 \leq q < r$. The proof is as follows: Egorov's theorem allows to choose $\Sigma \subset S$ such that $w_k \rightarrow w$ uniformly on Σ and $|S \setminus \Sigma| \leq \epsilon$ arbitrary small. By Hölder's inequality the remaining integral is estimated by $\int_{S \setminus \Sigma} |w_k - w|^q dx \leq \epsilon^{1-\frac{q}{r}} \|w_k - w\|_{L^r(S)}^q$.

Proof. Let us take a weakly convergent sequence $(v_k)_{k \in \mathbb{N}}$ in $K_{4,1}$ such that $v_k \rightharpoonup v$ in $H_{\text{cyl}}^1(r^3 dr dz)$ and $v_k \rightarrow v$ pointwise a.e. in Ω . By Lemma 6 one gets $v \in K_{4,1}$ and using Lemma 5 there exists a constant $C > 0$ such that

$$(2.10) \quad 0 \leq v_k(r, z), v(r, z) \leq C r^{-\frac{3}{2}} |z|^{-\frac{1}{2}} \text{ for all } k \in \mathbb{N} \text{ and almost all } (r, z) \in \Omega.$$

Our goal is now to show at least for a subsequence

$$(2.11) \quad \int_{\Omega} \frac{1}{r^2} F(r, z, r^2 v_k^2) r^3 d(r, z) \rightarrow \int_{\Omega} \frac{1}{r^2} F(r, z, r^2 v^2) r^3 d(r, z) \text{ as } k \rightarrow \infty$$

and

$$(2.12) \quad \int_{\Omega} f(r, z, r^2 v_k^2) v_k^2 r^3 d(r, z) \rightarrow \int_{\Omega} f(r, z, r^2 v^2) v^2 r^3 d(r, z) \text{ as } k \rightarrow \infty.$$

By (2.6) we find

$$\frac{1}{r^2} |F(r, z, r^2 v_k^2) - F(r, z, r^2 v^2)| r^3 \leq \epsilon r^2 (v_k^2 + v^2) r + C_{\epsilon} (|rv_k|^{p+1} + |rv|^{p+1}) r$$

and hence

$$(2.13) \quad (|F(r, z, r^2 v_k^2) - F(r, z, r^2 v^2)| - \epsilon r^2 (v_k^2 + v^2))^+ r \leq C_{\epsilon} (|rv_k|^{p+1} + |rv|^{p+1}) r.$$

Inspired by [11] and [12] the idea is to show

$$(2.14) \quad rv_k \rightarrow rv \text{ in } L^{p+1}(r dr dz) \text{ as } k \rightarrow \infty.$$

Once (2.14) is established we obtain a majorant $|rv_k|, |rv| \leq w \in L^{p+1}(r dr dz)$ (cf. Lemma A.1 in [19]). Together with (2.13) this majorant allows to apply Lebesgue's dominated convergence theorem and yields

$$(2.15) \quad \lim_{k \rightarrow \infty} \int_{\Omega} (|F(r, z, r^2 v_k^2) - F(r, z, r^2 v^2)| - \epsilon r^2 (v_k^2 + v^2))^+ r dr dz = 2\epsilon \|v\|_{L^2(r^3 dr dz)}^2.$$

If we set

$$a_k := \int_{\Omega} |F(r, z, r^2 v_k^2) - F(r, z, r^2 v^2)| r dr dz$$

and

$$b_k := \epsilon \|r^2 (v_k^2 + v^2)\|_{L^1(r dr dz)} = \epsilon (\|v_k\|_{L^2(r^3 dr dz)}^2 + \|v\|_{L^2(r^3 dr dz)}^2) \leq C\epsilon$$

then

$$\begin{aligned} \limsup_{k \in \mathbb{N}} a_k &\leq \limsup_{k \in \mathbb{N}} b_k + \limsup_{k \in \mathbb{N}} (a_k - b_k)^+ \\ &\leq C\epsilon + \limsup_{k \in \mathbb{N}} \left(\int_{\Omega} (|F(r, z, r^2 v_k^2) - F(r, z, r^2 v^2)| - \epsilon r^2 (v_k^2 + v^2))^+ r dr dz \right) \end{aligned}$$

$$\begin{aligned}
&\leq C\epsilon + \limsup_{k \in \mathbb{N}} \int_{\Omega} \left(|F(r, z, r^2 v_k^2) - F(r, z, r^2 v^2)| - \epsilon r^2 (v_k^2 + v^2) \right)^+ r dr dz \\
&\leq \epsilon (C + 2 \|v\|_{L^2(r^3 dr dz)}^2) \text{ by (2.15).}
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary this shows that $\lim_{k \rightarrow \infty} a_k = 0$ and therefore (2.11) holds. The proof of (2.12) is similar since $\left(f(r, z, r^2 v_k^2) r^2 v_k^2 - f(r, z, r^2 v^2) r^2 v^2 - \epsilon r^2 (v_k^2 + v^2) \right)^+ r$ satisfies an estimate just like (2.13) if we use (2.5) instead of (2.6).

It remains to prove (2.14). For this, we split our domain Ω into four parts $\Omega_1, \dots, \Omega_4$ and show (2.14) on each of these parts separately. The definitions of $\Omega_1, \dots, \Omega_4$ are as follows: For $R > 0$ let

$$\begin{aligned}
\Omega_1 &:= \{(r, z) \in \Omega : r < R, |z| < R\}, \quad \Omega_2 := \{(r, z) \in \Omega : r \geq R, |z| \geq R\}, \\
\Omega_3 &:= \{(r, z) \in \Omega : r < R, |z| \geq R\}, \quad \Omega_4 := \{(r, z) \in \Omega : r \geq R, |z| < R\}.
\end{aligned}$$

Convergence on Ω_1 : Follows from $rv_k \rightarrow rv$ in $L^q(K; r dr dz)$ for every compact subset $K \subset [0, \infty) \times \mathbb{R}$ and every $q \in [1, 6)$. This step works independently of the choice of $R > 0$.

Convergence on Ω_2 : Let $\varepsilon > 0$. With the help of (2.10) we calculate

$$\begin{aligned}
\int_{\Omega_2} |rv_k - rv|^{p+1} r dr dz &\leq 2^{p+1} \int_{\Omega_2} r^{p+1} (|v_k|^{p+1} + |v|^{p+1}) r dr dz \\
&\leq 2^{p+1} C^{p-1} \int_{\Omega_2} r^{-\frac{p-1}{2}} |z|^{-\frac{p-1}{2}} (|v_k(r, z)|^2 + |v(r, z)|^2) r^3 dr dz \\
&\leq C_1 \left(\|v_k\|_{H_{\text{cyl}}^1(r^3 dr dz)}^2 + \|v\|_{H_{\text{cyl}}^1(r^3 dr dz)}^2 \right) R^{-(p-1)} \leq C_2 R^{-(p-1)}
\end{aligned}$$

which is less or equal ε if we choose $R > 0$ large enough.

Convergence on Ω_3 : Due to symmetry in z -direction it is enough to focus on $\tilde{\Omega}_3 := \{(r, z) \in \Omega : r < R, z \geq R\}$. Let $\alpha > 0$ be arbitrary. Again by (2.10) we obtain

$$\{(r, z) \in \tilde{\Omega}_3 : v_k(r, z) > \alpha\} \subset \{(r, z) \in \tilde{\Omega}_3 : r z^{\frac{1}{3}} \leq C_\alpha\} =: S_\alpha,$$

where $C_\alpha = (C/\alpha)^{2/3}$ and C is the constant from (2.10). The set S_α has finite measure since

$$|S_\alpha| \leq \int_R^\infty \int_0^{C_\alpha z^{-1/3}} r^3 dr dz = \frac{C_\alpha^4}{4} \int_R^\infty z^{-\frac{4}{3}} dz = \frac{3}{4} C_\alpha^4 R^{-\frac{1}{3}} < \infty.$$

By the convergence principle from the remark above and since by (2.3) $\|rv_k\|_{L^6(r dr dz)} \leq \|v_k\|_{H_{\text{cyl}}^1(r^3 dr dz)}$ is bounded we obtain $\int_{S_\alpha} r^{p-1} |v_k - v|^{p+1} r^3 dr dz \rightarrow 0$ as $k \rightarrow \infty$ for $1 \leq p < 5$. It remains to prove the convergence on $\tilde{\Omega}_3 \setminus S_\alpha$. For almost all $(r, z) \in \tilde{\Omega}_3 \setminus S_\alpha$ we have that $v(r, z) = \lim_{k \rightarrow \infty} v_k(r, z) \leq \alpha$. Hence,

$$\int_{\tilde{\Omega}_3 \setminus S_\alpha} r^{p-1} |v_k - v|^{p+1} r^3 dr dz \leq R^{p-1} (2\alpha)^{p-1} \int_{\tilde{\Omega}_3 \setminus S_\alpha} |v_k - v|^2 r^3 dr dz \leq C \alpha^{p-1}.$$

In summary, since $\alpha > 0$ is arbitrary this shows (2.14) on Ω_3 .

Convergence on Ω_4 : Again it is enough to focus on $\tilde{\Omega}_4 := \{(r, z) \in \Omega : r \geq R, 0 \leq z < R\}$. Fix $z \in (0, R)$. Let us first show that

$$(2.16) \quad \int_{\{r \geq R\}} r^{p-1} |v_k(r, z) - v(r, z)|^{p+1} r^3 dr \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Since $v_k(r, \cdot)$ is nonincreasing in its last component we deduce

$$(2.17) \quad \int_0^\infty r^q v_k^q(r, z) r dr \leq \frac{1}{z} \int_0^z \int_0^\infty r^q v_k^q(r, \zeta) r d\zeta dr \leq \frac{1}{z} \int_\Omega r^q v_k^q(r, \zeta) r d(r, \zeta) \leq \frac{C}{z}$$

for all $q \in [2, 6]$ by (2.3). Thus for $q \in [2, 6]$ the sequence $\|\cdot v_k(\cdot, z)\|_{L^q((0, \infty), r dr)}$ is uniformly bounded in $k \in \mathbb{N}$. Moreover, (2.10) implies $v_k(r, z) \leq C(z)r^{-\frac{3}{2}}$ uniformly in $k \in \mathbb{N}$. Hence for $\tilde{R} > R$

$$\begin{aligned} \int_{\tilde{R}}^\infty r^{p-1} |v_k(r, z) - v(r, z)|^{p+1} r^3 dr &\leq (2C(z))^{p-1} \int_{\tilde{R}}^\infty r^{-\frac{p-1}{2}} |v_k(r, z) - v(r, z)|^2 r^3 dr \\ &\leq (2C(z))^{p-1} \tilde{R}^{\frac{1-p}{2}} \frac{C}{z} \text{ by (2.17).} \end{aligned}$$

The last term can be made arbitrarily small provided \tilde{R} is chosen big enough. To finish the proof of (2.16) it remains to prove $\int_R^{\tilde{R}} r^{p-1} |v_k(r, z) - v(r, z)|^{p+1} r^3 dr \rightarrow 0$ as $k \rightarrow \infty$. Since for almost all $z \in (0, R)$ we have $v_k(\cdot, z) \rightarrow v(\cdot, z)$ pointwise almost everywhere on (R, \tilde{R}) as well as the boundedness of $\|\cdot v_k(\cdot, z)\|_{L^6((0, \infty), r dr)}$ by (2.17) we can apply the convergence principle from the remark above and deduce

$$\int_R^{\tilde{R}} r^{p-1} |v_k(r, z) - v(r, z)|^{p+1} r^3 dr \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Hence (2.16) is accomplished for almost all $z \in (0, R)$.

Defining $\varphi_k(z) := \int_{\{r \geq R\}} r^{p-1} |v_k(r, z) - v(r, z)|^{p+1} r^3 dr$ we have $\varphi_k \rightarrow 0$ as $k \rightarrow \infty$ pointwise almost everywhere in $[0, R)$. The sequence $(\varphi_k)_{k \in \mathbb{N}}$ is bounded in $L^1([0, R), dz)$ since by (2.2)

$$\int_0^R \int_{\{r \geq R\}} r^{p-1} |v_k(r, z) - v(r, z)|^{p+1} r^3 dr dz \leq C \int_\Omega r^{p-1} (|v_k|^{p+1} + |v|^{p+1}) r^3 d(r, z) \leq \tilde{C}.$$

Moreover, for $p \in (1, 3]$, the sequence $(\varphi_k)_{k \in \mathbb{N}}$ is bounded in $W^{1,1}([0, R), dz)$ since

$$\begin{aligned} \left\| \frac{\partial \varphi_k}{\partial z} \right\|_{L^1([0, R], dz)}^2 &\leq \left(\int_0^R \int_R^\infty (p+1) r^{p-1} |v_k - v|^p \left| \frac{\partial v_k}{\partial z} - \frac{\partial v}{\partial z} \right| r^3 dr dz \right)^2 \\ &\leq \left(\int_\Omega (p+1) r^{p-1} |v_k - v|^p \left| \frac{\partial v_k}{\partial z} - \frac{\partial v}{\partial z} \right| r^3 d(r, z) \right)^2 \\ &\leq C \int_\Omega r^{2p-2} |v_k - v|^{2p} r^3 d(r, z) \int_\Omega \left| \frac{\partial v_k}{\partial z} - \frac{\partial v}{\partial z} \right|^2 r^3 d(r, z) \\ &= C \|r(v_k - v)\|_{L^{2p}(rdrdz)}^{2p} \int_\Omega \left| \frac{\partial v_k}{\partial z} - \frac{\partial v}{\partial z} \right|^2 r^3 d(r, z) \leq C. \end{aligned}$$

Hence, by the compact embedding $W^{1,1}([0, R), dz) \hookrightarrow L^1([0, R), dz)$ we conclude that at least a subsequence of $(\varphi_k)_{k \in \mathbb{N}}$ is converging in $L^1([0, R), dz)$ to a limit function, which must be 0 since we have already asserted the pointwise a.e. convergence to 0 on $[0, R)$. This shows (2.14) on Ω_4 for $p \in (1, 3]$. For $p \in (3, 5)$ we make use of Hölder's interpolation, namely,

$$\|rv_k - rv\|_{L_{\text{cyl}}^{p+1}(\Omega_4, r dr dz)}^{p+1} \leq \|rv_k - rv\|_{L_{\text{cyl}}^4(\Omega_4, r dr dz)}^{4\theta} \|rv_k - rv\|_{L_{\text{cyl}}^6(\Omega_4, r dr dz)}^{6(1-\theta)} \leq \tilde{C} \|rv_k - rv\|_{L_{\text{cyl}}^4(\Omega_4, r dr dz)}^{4\theta} \rightarrow 0$$

as $k \rightarrow \infty$, where $\theta \in (0, 1)$ is chosen such that $p+1 = 4\theta + 6(1-\theta)$, i.e., $\theta = \frac{5-p}{2}$.

The combination of convergences on $\Omega_1, \dots, \Omega_4$ finally proves (2.14). \square

For our last lemma we need the notion of cylindrical C_c^∞ -functions which we introduce now.

Definition 8. A function $u = u(r, z)$ belongs to $C_c^\infty([0, \infty) \times \mathbb{R})$ if and only if $u \in C^\infty([0, \infty) \times \mathbb{R})$, $\text{supp } u$ is compact in $[0, \infty) \times \mathbb{R}$ and $\frac{\partial^j u}{\partial r^j}(0, z) = 0$ for all odd integers $j \in 2\mathbb{N} - 1$.

Remark: Since $u \in C_c^\infty([0, \infty) \times \mathbb{R})$ is equivalent to $\tilde{u} \in C_c^\infty(\mathbb{R}^5)$ with $\tilde{u}(x) := u(|(x_1, \dots, x_4)|, x_5)$ we see that $C_c^\infty([0, \infty) \times \mathbb{R})$ is dense in $H_{\text{cyl}}^1(r^3 dr dz)$.

Lemma 9. For $u \in H_{\text{cyl}}^1(r^3 dr dz)$ we have $\|u^\star\| \leq \|u\|$ where \star denotes Steiner-symmetrization with respect to z and $\|\cdot\|$ is the equivalent norm from Theorem 1. Moreover

$$I(u) \leq I(u^\star) \quad \text{and} \quad I'(u)[u] \leq I'(u^\star)[u^\star].$$

Proof. We begin by recalling several classical rearrangement inequalities from [9], [10]. Recall first the Pólya-Szegő inequality

$$(2.18) \quad \int_{\mathbb{R}^n} |\nabla f^\circ|^2 dx \leq \int_{\mathbb{R}^n} |\nabla f|^2 dx$$

for $f \in H^1(\mathbb{R}^n)$ and $^\circ$ denoting Schwarz-symmetrization (also called symmetrically decreasing rearrangement). Furthermore we have for $0 \leq f, g \in L^2(\mathbb{R}^n)$ the classical rearrangement inequality

$$(2.19) \quad \int_{\mathbb{R}} f g dx \leq \int_{\mathbb{R}} f^\circ g^\circ dx$$

and the nonexpansivity of rearrangement

$$(2.20) \quad \int_{\mathbb{R}^n} |f^\circ - g^\circ|^2 dx \leq \int_{\mathbb{R}^n} |f - g|^2 dx.$$

From (2.18) we immediately receive for $u \in H_{\text{cyl}}^1(r^3 dr dz)$ that

$$(2.21) \quad \int_{\mathbb{R}} |\nabla_z u^\star|^2 dz \leq \int_{\mathbb{R}} |\nabla_z u|^2 dz.$$

Next we want to establish a similar inequality for $\nabla_r u$. We do this first for $u \in C_c^\infty([0, \infty) \times \mathbb{R})$. With the help of (2.20) we find that

$$\int_{\mathbb{R}} \left| \frac{u^\star(r+t, z) - u^\star(r, z)}{t} \right|^2 dz \leq \int_{\mathbb{R}} \left| \frac{u(r+t, z) - u(r, z)}{t} \right|^2 dz$$

for almost all $r, t \in [0, \infty)$. Sending $t \rightarrow 0$ and using Fatou's lemma on the left side of the inequality yields

$$(2.22) \quad \int_{\mathbb{R}} |\nabla_r u^\star|^2 dz \leq \int_{\mathbb{R}} |\nabla_r u|^2 dz$$

for $u \in C_c^\infty([0, \infty) \times \mathbb{R})$ and almost all $r \in [0, \infty)$. Since Steiner Symmetrization is continuous in H^1 (see Theorem 1 in [7]) we obtain by approximation that (2.22) is indeed valid for all $u \in H_{\text{cyl}}^1(r^3 dr dz)$.

Together with (2.21) we obtain $\int_{\mathbb{R}} |\nabla_{r,z} u^\star|^2 dz \leq \int_{\mathbb{R}} |\nabla_{r,z} u|^2 dz$ for almost all $r \geq 0$ and integration leads to

$$(2.23) \quad \int_{\mathbb{R}} \int_0^\infty |\nabla_{r,z} u^\star|^2 r^3 dr dz \leq \int_{\mathbb{R}} \int_0^\infty |\nabla_{r,z} u|^2 r^3 dr dz.$$

Fixing $r \in [0, \infty)$ and applying (2.19) to $f(\cdot) = \text{ess sup } V - V(r, \cdot)$ and $g(\cdot) = u^2(r, \cdot)$ gives

$$\begin{aligned} \int_{\mathbb{R}} (\text{ess sup } V - V(r, \cdot)) u^2(r, \cdot) dz &\leq \int_{\mathbb{R}} (\text{ess sup } V - V(r, \cdot))^\star (u^2)^\star(r, \cdot) dz \\ &= \int_{\mathbb{R}} (\text{ess sup } V - V(r, \cdot)) (u^\star)^2(r, \cdot) dz. \end{aligned}$$

Using $\|u(r, \cdot)\|_{L^2(\mathbb{R})} = \|u^\star(r, \cdot)\|_{L^2(\mathbb{R})}$ this results in

$$(2.24) \quad \int_{\mathbb{R}} \int_0^\infty V(r, z) (u^\star)^2 r^3 dr dz \leq \int_{\mathbb{R}} \int_0^\infty V(r, z) u^2 r^3 dr dz.$$

The combination of (2.23) and (2.24) yields the claimed inequality $\|u^\star\|^2 \leq \|u\|^2$.

Assumption (v) on f allows to apply Theorem 5.1 in [6] and to deduce

$$(2.25) \quad I'(u)[u] = \int_{\Omega} f(r, z, r^2 u^2) u^2 r^3 d(r, z) \leq \int_{\Omega} f(r, z, r^2 u^{\star 2}) u^{\star 2} r^3 d(r, z) = I'(u^\star)[u^\star].$$

Moreover, using (v) with $s = 0$ shows that for all $r \in [0, \infty)$, $\sigma \geq 0$ the function $z \mapsto f(r, z, \sigma^2)$ is symmetrically nonincreasing in z and hence

$$\Phi_\sigma(r, z, s) := F(r, z, r^2(s + \sigma)^2) - F(r, z, r^2 s^2) = \int_{r^2 s^2}^{r^2(s+\sigma)^2} f(r, z, t) dt$$

is symmetrically nonincreasing in z . Applying once more Theorem 5.1 in [6] yields

$$I(u) = \int_{\Omega} \frac{1}{2r^2} F(r, z, r^2 u^2) r^3 d(r, z) \leq \int_{\Omega} \frac{1}{2r^2} F(r, z, r^2 u^{\star 2}) r^3 d(r, z) = I(u^\star).$$

This finishes the proof of the lemma. \square

3. PROOF OF THEOREM 1

Proof. Recall from Lemma 7 the definition $I(u) := \int_{\Omega} \frac{1}{2r^2} F(r, z, r^2 u^2) r^3 d(r, z)$ for $u \in H_{\text{cyl}}^1(r^3 dr dz)$. We show that the assumptions (i)-(iii) of Theorem 12 in [18] are satisfied. Let $\varepsilon > 0$. The growth assumptions (i) and (ii) on f imply that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that the global estimate $0 \leq f(r, z, s) \leq \epsilon + C_\epsilon |s|^{\frac{p-1}{2}}$ holds. Together with (2.2) we obtain

$$\begin{aligned} |I'(u)[v]| &= \left| \int_{\Omega} f(r, z, r^2 u^2) u v r^3 d(r, z) \right| \\ &\leq \varepsilon \int_{\Omega} |ru| |rv| r d(r, z) + C_\epsilon \int_{\Omega} |ru|^p |rv| r d(r, z) \\ &\leq \varepsilon C \|u\|_{H_{\text{cyl}}^1(r^3 dr dz)} \|v\|_{H_{\text{cyl}}^1(r^3 dr dz)} + \tilde{C}_\epsilon \|u\|_{H_{\text{cyl}}^1(r^3 dr dz)}^p \|v\|_{H_{\text{cyl}}^1(r^3 dr dz)} \end{aligned}$$

Taking the supremum over all $v \in H_{\text{cyl}}^1(r^3 dr dz)$ with $\|v\|_{H_{\text{cyl}}^1(r^3 dr dz)} = 1$ we see that

$$(3.1) \quad I'(u) = o(\|u\|) \text{ as } u \rightarrow 0.$$

Moreover, due to assumption (iii) on f the map

$$(3.2) \quad s \mapsto \frac{I(su)[u]}{s} = \int_{\Omega} f(r, z, s^2 r^2 u^2) u^2 r^3 d(r, z) \text{ is strictly increasing for all } u \neq 0 \text{ and } s > 0.$$

Next we claim that

$$(3.3) \quad \frac{I(su)}{s^2} \rightarrow \infty \text{ as } s \rightarrow \infty \text{ uniformly for } u \text{ on weakly compact subsets } W \text{ of } H_{\text{cyl}}^1(r^3 dr dz) \setminus \{0\}.$$

Suppose not. Then there are $(u_k)_{k \in \mathbb{N}} \subset W$ and $s_k \rightarrow \infty$ as $k \rightarrow \infty$ such that $\frac{I(s_k u_k)}{s_k^2}$ is bounded as $k \rightarrow \infty$. But along a subsequence we have $u_k \rightharpoonup u \neq 0$ and $u_k(x) \rightarrow u(x)$ pointwise almost everywhere. Let

$\Omega^\# := \{(r, z) \in \Omega : u(r, z) \neq 0\}$. Then $|\Omega^\#| > 0$ and on $\Omega^\#$ we have $|s_k u_k(r, z)| \rightarrow \infty$ as $k \rightarrow \infty$. Fatou's lemma and assumption (iv) on F imply

$$\frac{I(s_k u_k)}{s_k^2} = \int_{\Omega} \frac{F(r, z, s_k^2 r^2 u_k^2)}{2s_k^2 r^2} r^3 d(r, z) \geq \int_{\Omega^\#} \frac{F(r, z, s_k^2 r^2 u_k^2)}{2s_k^2 r^2 u_k^2} u_k^2 r^3 d(r, z) \rightarrow \infty \text{ as } k \rightarrow \infty,$$

a contradiction. In summary, (3.1), (3.2), (3.3) imply that (i)-(iii) of Theorem 12 in [18] are satisfied.

Now we take a sequence $(u_k)_{k \in \mathbb{N}} \subset M$ such that $J(u_k) \rightarrow \inf_M J$ as $k \rightarrow \infty$. Since $\|\nabla_{r,z} |u_k|\|_{L^2} = \|\nabla_{r,z} u_k\|_{L^2}$ we can assume that $u_k \geq 0$ for all $k \in \mathbb{N}$. Then Theorem 12 in [18] guarantees that for every k there is a unique $t_k > 0$ such that $v_k := t_k u_k^* \in M$. We show next that $t_k \leq 1$ for all $k \in \mathbb{N}$. Assume $t_k > 1$. Then

$$\begin{aligned} \int_{\Omega} f(r, z, r^2 u_k^{*2}) u_k^{*2} r^3 d(r, z) &< \int_{\Omega} f(r, z, t_k^2 r^2 u_k^{*2}) u_k^{*2} r^3 d(r, z) \quad \text{by assumption (iii)} \\ &= \|u_k^*\|^2 \quad \text{since } t_k u_k^* \in M \\ &\leq \|u_k\|^2 \quad \text{by Lemma 9} \\ &= \int_{\Omega} f(r, z, r^2 u_k^2) u_k^2 r^3 d(r, z) \quad \text{since } u_k \in M. \end{aligned}$$

This contradicts the inequality $I'(u_k)[u_k] \leq I'(u_k^*)[u_k^*]$ from Lemma 9 and thus $t_k \leq 1$ for all $k \in \mathbb{N}$.

Next notice that for fixed $(r, z, s) \in [0, \infty) \times \mathbb{R} \times [0, \infty)$ and $t \in (0, 1]$ one has

$$\frac{d}{dt} (t^2 f(r, z, s^2) s^2 - F(r, z, t^2 s^2)) = 2ts^2 (f(r, z, s^2) - f(r, z, t^2 s^2)) > 0$$

since f is strictly increasing in its last variable by assumption (iii). This shows that the map $t \mapsto t^2 f(r, z, s^2) s^2 - F(r, z, t^2 s^2)$ is strictly increasing for $t \in [0, 1]$. From this monotonicity and the inequality $I(t_k u_k) \leq I(t_k u_k^*)$ from Lemma 9 we conclude

$$\begin{aligned} 2J(v_k) &= \int_{\Omega} \left(t_k^2 |\nabla_{r,z} u_k^*|^2 + V(r, z) t_k^2 u_k^{*2} - \frac{1}{r^2} F(r, z, r^2 t_k^2 u_k^{*2}) \right) r^3 d(r, z) \\ &\leq \int_{\Omega} \left(t_k^2 |\nabla_{r,z} u_k|^2 + V(r, z) t_k^2 u_k^2 - \frac{1}{r^2} F(r, z, r^2 t_k^2 u_k^2) \right) r^3 d(r, z) \\ (3.4) \quad &= \int_{\Omega} \frac{1}{r^2} (f(r, z, r^2 u_k^2) t_k^2 r^2 u_k^2 - F(r, z, r^2 t_k^2 u_k^2)) r^3 d(r, z) \\ &\leq \int_{\Omega} \frac{1}{r^2} (f(r, z, r^2 u_k^2) r^2 u_k^2 - F(r, z, r^2 u_k^2)) r^3 d(r, z) \\ &= 2J(u_k). \end{aligned}$$

So $(v_k)_{k \in \mathbb{N}} \subset M$ is also a minimizing sequence for J which belongs to $K_{4,1}$. The boundedness of $(v_k)_{k \in \mathbb{N}}$ is established in Proposition 14 in [18]. Hence, we find $v_\infty \in H_{\text{cyl}}^1(r^3 dr dz)$ such that $v_k \rightharpoonup v_\infty$ in $H_{\text{cyl}}^1(r^3 dr dz)$ along a subsequence as $k \rightarrow \infty$. In addition, $v_\infty \in K_{4,1}$ due to Lemma 6 and $v_\infty \neq 0$ by Proposition 14 in [18] where instead of the weak sequential continuity of I on all of $H_{\text{cyl}}^1(r^3 dr dz)$ we use it only on $K_{4,1}$ as stated in Lemma 7.

Let us show that $v_\infty \in M$. Since $v_\infty \neq 0$ we can choose $t_\infty > 0$ such that $t_\infty v_\infty \in M$. In the same manner as before for the sequence t_k we can show that $t_\infty \leq 1$. Assume $t_\infty < 1$. Then as in (3.4) and using the weak sequential continuity on $K_{4,1}$ as shown in Lemma 7 we find

$$2J(t_\infty v_\infty) < \int_{\Omega} \frac{1}{r^2} (f(r, z, r^2 v_\infty^2) r^2 v_\infty^2 - F(r, z, r^2 v_\infty^2)) r^3 d(r, z)$$

$$\begin{aligned}
&= \lim_{k \rightarrow \infty} \int_{\Omega} \frac{1}{r^2} \left(f(r, z, r^2 v_k^2) r^2 v_k^2 - F(r, z, r^2 v_k^2) \right) r^3 d(r, z) \\
&= 2 \inf_M J \leq 2J(t_{\infty} v_{\infty})
\end{aligned}$$

which is a contradiction. So $t_{\infty} = 1$ and thus $v_{\infty} \in M$. Then by the weak lower semi-continuity of $\|\cdot\|$ and once again the weak sequential continuity of I we conclude

$$J(v_{\infty}) \leq \liminf_{k \rightarrow \infty} J(v_k) = \inf_M J \leq J(v_{\infty}).$$

Hence, $v_{\infty} \in K_{4,1}$ is a minimizer of J on M , i.e., a ground state of (1.3) which is Steiner symmetric in z with respect to $\{z = 0\}$. \square

APPENDIX

Here we prove that the condition $V \geq 0$ and $\inf_{B_R^c} V > 0$ for some $R > 0$ implies that on $H_{\text{cyl}}^1(r^3 dr dz)$ the expression $\left(\int_{\Omega} (|\nabla_{r,z} u|^2 + V(r, z) u^2) r^3 d(r, z) \right)^{\frac{1}{2}}$ is an equivalent norm. Suppose not. Then there is a sequence $(u_k)_{k \in \mathbb{N}}$ such that $\|u_k\|_{L^2(r^3 dr dz)} = 1$ and $\int_{\Omega} (|\nabla_{r,z} u_k|^2 + V(r, z) u_k^2) r^3 d(r, z) \rightarrow 0$ as $k \rightarrow \infty$. In particular,

$$(3.5) \quad \int_{\Omega} |\nabla_{r,z} u_k|^2 r^3 d(r, z) \rightarrow 0 \text{ and } \int_{B_R^c} u_k^2 r^3 d(r, z) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Let χ denote a smooth cut-off function such that $\chi(r, z) = 1$ for $0 \leq \sqrt{r^2 + z^2} < R$ and $\chi(r, z) = 0$ for $\sqrt{r^2 + z^2} \geq R + 1$. Then $v_k := \chi u_k \in H_{0, \text{cyl}}^1(B_{R+1}, r^3 dr dz)$ and

$$|\nabla_{r,z} v_k|^2 = \chi^2 |\nabla_{r,z} u_k|^2 + |\nabla_{r,z} \chi|^2 u_k^2 + 2u_k \chi \nabla_{r,z} u_k \cdot \nabla_{r,z} \chi.$$

Hence, by (3.5)

$$\begin{aligned}
(3.6) \quad \int_{\Omega} |\nabla_{r,z} v_k|^2 r^3 d(r, z) &\leq 2 \int_{\Omega} \chi^2 |\nabla_{r,z} u_k|^2 r^3 d(r, z) + 2 \int_{\Omega} u_k^2 |\nabla_{r,z} \chi|^2 r^3 d(r, z) \\
&\leq 2 \int_{\Omega} |\nabla_{r,z} u_k|^2 r^3 d(r, z) + 2 \|\nabla_{r,z} \chi\|_{\infty}^2 \int_{B_{R+1} \setminus B_R} u_k^2 r^3 d(r, z) \rightarrow 0 \text{ as } k \rightarrow \infty.
\end{aligned}$$

In particular, $\int_{B_{R+1}} |\nabla_{r,z} v_k|^2 r^3 d(r, z) \rightarrow 0$ as $k \rightarrow \infty$. By Poincaré's inequality, $\|u_k\|_{L^2(r^3 dr dz)} = 1$ and (3.5) we see

$$C_P \int_{B_{R+1}} |\nabla_{r,z} v_k|^2 r^3 d(r, z) \geq \int_{B_{R+1}} v_k^2 r^3 d(r, z) \geq \int_{B_R} u_k^2 r^3 d(r, z) = 1 - o(1),$$

contradicting (3.6). \square

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REFERENCES

- [1] Azzollini, A., Benci, V., D'Aprile, T. and Fortunato, D.: Existence of static solutions of the semilinear Maxwell equations. *Ricerche di matematica*, 55(2), 123-137, (2006).
- [2] Bartsch, T., Dohnal, T., Plum, M. and Reichel, W.: Ground states of a nonlinear curl-curl problem in cylindrically symmetric media. arXiv preprint arXiv:1411.7153, (2014).
- [3] Bartsch, T. and Mederski, J.: Ground and bound state solutions of semilinear time-harmonic Maxwell equations in a bounded domains. *Arch. Ration. Mech. Anal.*, 215(1), 283–306, (2015).
- [4] Bartsch, T. and Mederski, J.: Nonlinear time-harmonic Maxwell equations in an anisotropic bounded medium. arXiv preprint arXiv:1509.01994, (2015).
- [5] Benci, V. and Fortunato, D.: Towards a unified field theory for classical electrodynamics. *Arch. Ration. Mech. Anal.*, 173(3), 379–414, (2004).
- [6] Brock, F.: Continuous rearrangement and symmetry of solutions of elliptic problems. *Proceedings of the Indian Academy of Sciences-Mathematical Sciences*. Vol. 110. No. 2. Springer India, (2000).
- [7] Burchard, A.: Steiner symmetrization is continuous in $W^{1,p}$. *Geometric & Functional Analysis GAFA* 7.5: 823-860, (1997).
- [8] D'Aprile, T. and Siciliano, G.: Magnetostatic solutions for a semilinear perturbation of the Maxwell equations. *Adv. Differential Equations*, 16(5-6), 435–466, (2011).
- [9] Lieb, E. H.: Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation. *Studies in Applied Mathematics* 57: 93-105, (1977).
- [10] Lieb, E. H. and Loss, M.: *Analysis*, volume 14 of graduate studies in mathematics. American Mathematical Society, Providence, RI, 4, (2001).
- [11] Lions, P.-L.: Minimization problems in $L^1(\mathbb{R}^3)$. *Journal of Functional Analysis*, 41(2), 236-275, (1981).
- [12] Lions, P.-L.: Symétrie et compacité dans les espaces de Sobolev. *Journal of Functional Analysis* 49.3: 315-334, (1982).
- [13] Liu, S.: On superlinear problems without the Ambrosetti and Rabinowitz condition. *Nonlinear Analysis: Theory, Methods & Applications*, 73(3), 788-795, (2010).
- [14] Mederski, J.: Ground states of time-harmonic semilinear Maxwell equations in \mathbb{R}^3 with vanishing permittivity. arXiv preprint arXiv:1406.4535, (2014), to appear in *Arch. Ration. Mech. Anal.*
- [15] Nehari, Z.: On a class of nonlinear second-order differential equations. *Transactions of the American Mathematical Society* 95.1: 101-123, (1960).
- [16] Nehari, Z.: Characteristic values associated with a class of nonlinear second-order differential equations. *Acta Mathematica* 105.3: 141-175, (1961).
- [17] Stuart, C. A.: A variational approach to bifurcation in L^p on an unbounded symmetrical domain. *Mathematische Annalen*, 263(1), 51-59, (1983).
- [18] Szulkin, A. and Weth, T.: The method of Nehari manifold. *Handbook of nonconvex analysis and applications*: 597-632, (2010).
- [19] Willem, M.: *Minimax theorems*. Vol. 24. Springer Science & Business Media, (1997).

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